29/6/2015

9 am

Prof. Gopal Basak

Probability Theory - lecture 1

Let Ω be the sample space and $\mathcal{A}$ be the power set of Ω

Then probability is a set function s.t

$$P : \mathcal{A} \longrightarrow \mathbb{R}^+ \text{ satisfying}$$

(i) $P(\phi) = 0$, $P(\Omega) = 1$

(ii) $A \subseteq B \Rightarrow P(A) \subseteq P(B)$ where $A, B \in \mathcal{A}$

(iii) For $\{A_n\}_{n \geq 1} \in \mathcal{A}$

$$P(\cup A_n) = \sum_{n \geq 1} P(A_n) \text{ whenever } \{A_n\}\text{s are disjoint.}$$

Q.1. Verify that the quadruplet $\{a, b, c, d\}$ satisfy the ~~propo~~ probability condition where

$$\Omega = \{HH, HT, TH, TT\}$$

$$\mathcal{A} = \{\underbrace{\{HH\}}_{a \geq 0}, \underbrace{\{HT\}}_{b \geq 0}, \underbrace{\{TH\}}_{c \geq 0}, \underbrace{\{TT\}}_{d \geq 0}, \dots\}$$

s.t $a+b+c+d = 1$.

Independence

If $(\Omega, \mathcal{A}, P)$ is a probability space, for $A, B \in \mathcal{A}$, A & B are independent if

$$P(A \cap B) = P(A)\, P(B).$$

The assignment

$a = pq$

$b = p(1-q)$

$c = (1-p)q$

$d = (1-p)(1-q)$

is a generalised way of assignment of probabilities in 2 coin toss case.

where p = Probability of first coin being head

q = " " second "

<u>σ-field</u>

Let Ω be the whole set

Then a collection of subsets $\mathcal{A}$ is called a σ-field if

(i) $\phi, \Omega \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) $\{A_n\}_{n\geq 1} \in \mathcal{A}$ then $\bigcup_{n\geq 1} A_n \in \mathcal{A}$ & $\bigcap_{n\geq 1} A_n \in \mathcal{A}$

eg. 1. $\{\phi, \Omega\}$

2. $\{\phi, A, A^c, \Omega\}$.

3. If A & B are given, then partition them into disjoint sets $\{A \setminus B, B \setminus A, A \cap B, (A \cup B)^c\}$ and then use this partition.

Q.2. Let $\{A_1, \dots, A_k\}$ be the partition of Ω. Then find the σ-field generated by $\{A_1, \dots, A_k\}$.

Defn : Let C be a collection of subsets of Ω. Then σ-field generated by $\mathcal{C}$ means $\bigcap_{\mathcal{A} \supseteq \mathcal{C}} \mathcal{A}$ and is denoted by $\sigma(\mathcal{C})$.

Q.3. σ-field is either finite or uncountable.

Properties

(a) Let $\mathcal{A}_1$ & $\mathcal{A}_2$ be two σ-fields. Then $\mathcal{A}_1 \cap \mathcal{A}_2$ is a σ-field but $\mathcal{A}_1 \cup \mathcal{A}_2$ need not be.

(b) Let $\{\mathcal{A}_n\}$ be the sequence of σ-fields s.t $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ $\forall\, n \geq 1$.

Then $\bigcup_{n \geq 1} \mathcal{A}_n$ need not be a σ-field even though countable unions are σ-fields.

eg. $\mathcal{A}_n = \sigma\left(\left(0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \dots \left(\frac{2^n-1}{2^n}, 1\right]\right)$.

then $\{1\} \notin \mathcal{A}_i$ for any i $\therefore$ $\{1\} \notin \bigcup_{n \geq 1} \mathcal{A}_n$

But $\{1\} = \bigcap_{n \geq 1} \left(\frac{2^n-1}{2^n}, 1\right]$ and hence should belong to $\bigcup_{n \geq 1} \mathcal{A}_n$ if $\bigcup_{n \geq 1} \mathcal{A}_n$ is a σ-field.

Atom : A is an atom if for any $B \in \mathcal{A}$ & $B \subseteq A$ then $B = A$ or $B = \phi$.

eg. $\Omega = \{1, 2, \ldots\}$
$\mathcal{A}_n = \sigma(\{1\}, \{2\}, \ldots, \{n\})$

Then set of odd nos. $\notin \bigcup_{n \geq 1} \mathcal{A}_n$

& hence $\bigcup_{n \geq 1} \mathcal{A}_n$ is not a σ-field.

Borel σ-field

It is the σ-field generated by open sets

Consider $\mathcal{A} = \sigma(\{[a, b] : a, b \text{ rationals}\})$.

Atoms are all singletons

Let $\mathcal{A}_a$ = σ-field generated by singletons in $\mathbb{R}$
= {all countable & co-countable subsets in $\mathbb{R}$}.

eg. (i) $P((a, b]) = b - a$
(ii) $P((a, b]) = \int_a^b f(x)\,dx$

for any $f \geq 0$ & $\int_a^b f(x)\,dx < \infty$ for any $a, b \in \mathbb{R}$

These are examples where measure at a point is 0 but measure in an interval is non-zero.

For probability, $\int_a^b f(x)\,dx = 1$.

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Probability Theory - Lecture 2

Q.5. $\mathcal{C} = \{A_1, A_2, \dots\}$

Then $\sigma(\mathcal{C})$ contains an atom which is given by $\{\bigcap_{i\geq 1} D_i : D_i = A_i \text{ or } A_i^c\}$.

Consider $\mathcal{C} = \{$finite disjoint union of $(a,b]$, a, b rea

Then $P(\bigcup_i (a_i, b_i]) = \sum_i \int_{a_i}^{b_i} f(x)\,dx$.

Then $\mathcal{C}$ is a field, i.e

(i) $\phi, \Omega \in \mathcal{C}$

(ii) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$

(iii) $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ & $A \cup B \in \mathcal{C}$

(the only point of difference with σ-fields being the non-existence of countable union in field).

$F(b) \to 1$ as $b \to \infty$

& $F(a) \to 0$ as $a \to -\infty$

i.e $\forall\ \varepsilon > 0$, $\exists$ a, b reals st.

$1 - F(b) < \varepsilon$

& $F(a) < \varepsilon$.

Step 1 : First we need to show that P is countably additive on $\mathcal{C}$.

Step 2 : Extend P to monotone limits of $\mathcal{C}$, say to $\mathcal{G}$ where

$$\mathcal{G} = \left\{ A : A = \lim A_n \text{ whenever } A_n \uparrow A \text{ or } A_n \downarrow A \text{ for } \{A_n\} \in \mathcal{C} \right\}.$$

Step 3 : $P^*(A) = \inf \left\{ P\left(\bigcup_{i \geq 1} A_i\right) : \bigcup A_i \supseteq A \ \& \ \bigcup A_i \in \mathcal{G} \right\}$,

(i) P^* is countably subadditive. (i.e P^* of union $\leq$ sum of P^*).

(ii) $P^*\Big|_{\mathcal{C} \text{ or } \mathcal{G}} = P$

(iii) $P^*(A \cup A^c) = P^*(A) + P^*(A^c) \overset{?}{=} 1$

Q6. Show that if $\mathcal{M}$ is a monotone class containing $\mathcal{G}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{M}$

Step 4 : Consider $\mathcal{F} = \{ A \subseteq \Omega : P^*(A) + P^*(A^c) = 1 \}$.

Show that $\mathcal{F}$ is a field.

Step 5 : p^* is finite additive on $\mathcal{F}$.

Step 6 : p^* is countably additive on $\mathcal{F}$

Step 7 : $\mathcal{F}$ is a σ-field and hence $\sigma(\mathcal{C}) \subseteq \mathcal{F}$.

Step 8 : ~~$|\mathcal{F}| > |R|$ $|\sigma(\mathcal{C})| = |R|$.~~

If $A \in \mathcal{F}$ & $p^*(A) = 0$ then $B \in \mathcal{F}$ for any $B \subseteq A$ & $p^*(B) = 0$

$\therefore$ $|\mathcal{F}| > |R|$ & $|\sigma(\mathcal{C})| = |R|$

$\therefore$ $|\mathcal{F}| = 2^{|R|}$ ($\because$ there is nothing in between $|R|$ & $2^{|R|}$)

Q7. $\mathcal{C} = \{(a, b] : a, b \text{ rationals}\}$

$\mathcal{C}_1 = \{$ countable unions & compliments of sets of $\mathcal{C}\}$

$\mathcal{C}_2 = \{$ " " " $\mathcal{C}_1\}$

$\vdots$

We keep doing this till ~~β where β~~ α (α being the first uncountable ordinal)

Then $\sigma(\mathcal{C}) = \bigcup\limits_{\beta < \alpha} \mathcal{C}_\beta$ is the smallest σ-field.

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Probability Theory - Lecture 3

$$P(A) = \int_A dP = \int_\Omega 1_A \, dP$$

If $f = \sum_{i=1}^{n} c_i 1_{A_i} = \{ c_i$ if $x \in A_i$

where $\{A_i\}$ are partitions of Ω

Then, $\int_\Omega f \, dP = \int_\Omega \sum c_i 1_{A_i} \, dP$

$$= \sum_{i=1}^{n} c_i \int_\Omega 1_{A_i} \, dP$$

$$= \sum c_i P(A_i)$$

Measurable functions

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

$f : \Omega \longrightarrow \mathbb{R}$

If $f^{-1}(B) \in \mathcal{F}_i \quad \forall \; B \in \mathcal{B}(\mathbb{R})$

Then f is called a measurable function

But showing for every Borel set is difficult

$\therefore$ Consider $\mathcal{C} = \{ B \in \mathcal{B}(\mathbb{R}) : f^{-1}(B) \in \mathcal{F} \}$.

and show that $\mathcal{C}$ is a σ-field.

ϕ, $\mathbb{R} \in \mathcal{C}$

$A \in \mathcal{C} \Rightarrow f^{-1}(A^c) = [f^{-1}(A)]^c \in \mathcal{F} \Rightarrow A^c \in \mathcal{C}$

$$f^{-1}\left(\bigcup_n A_n\right) = \bigcup_n f^{-1}(A_n) \in \mathcal{F}$$

$$\Rightarrow \bigcup_n A_n \in \mathcal{C}$$

$$f^{-1}\left(\bigcap_n A_n\right) = \bigcap_n f^{-1}(A_n) \in \mathcal{F}$$

$$\Rightarrow \bigcap_n A_n \in \mathcal{C}$$

$\therefore$ $\mathcal{C}$ is a σ-field.

But $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R})$

Also $\{(-\infty, x] : x \in \mathbb{R}\} \subseteq \mathcal{C}$

$\therefore$ $\sigma\{(-\infty, x] : x \in \mathbb{R}\} \subseteq \sigma(\mathcal{C}) = \mathcal{C}$

$\Rightarrow$ $\mathcal{B}(\mathbb{R}) = \mathcal{C}$

$\therefore$ Checking for measurability reduces to checking if $f^{-1}(-\infty, x] \in \mathcal{F}$ $\forall$ $x \in \mathbb{R}$.

Let f be a measurable function, $f \geq 0$.

Define $\int_\Omega f \, dP = \sup \left\{ \int_\Omega s \, dP : 0 \leq s \leq f, \ s \text{ being a simple function} \right\}$

<u>Defn</u>: Simple functions are those that take finitely many values on partitions of Ω.

Q. $\int_\Omega (f+g)\,dP = \int_\Omega f\,dP + \int_\Omega g\,dP$ wherever they are defined.

$$\int_\Omega cf\,dP = c\int_\Omega f\,dP$$

Case 1: Let $f \geq 0$ be a measurable function & bounded.

Define

$$f_n = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \\ n, & f(x) \geq n \end{cases}$$

$$\therefore \int_\Omega f_n\,dP = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\left(f^{-1}\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right) + nP(f^{-1}[n,\infty)).$$

Then $f_n \uparrow$ in n

& $f - f_n \leq \frac{1}{2^n} \longrightarrow 0$ for large enough n

$n > \sup |f(x)|.$

Case 2: Let f be a measurable function & f need not be bounded.

Let $f_M = f \wedge M$ where $M \geq 0$ is a constant.

For each x,

$|f_M - f| \to 0$ as $M \to \infty$

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$$\therefore \int f_n \, dP \to \int f \, dP$$

Monotone Convergence Theorem

Q. Let $f_n \geq 0$ be measurable functions. $f_n \uparrow f$. Show that (i) f is measurable.
(ii) lim sup & lim inf are ~~measurable~~ measurable
(iii) limit (if exists) is measurable.

Let $f_n \geq 0$ be measurable function & $f_n \uparrow f$. Then

$$\int f \, dP = \lim_{n \to \infty} \int f_n \, dP$$

Case 3 : For f measurable,
take $f = f^{+} - f^{-1}$

$$f^{+} = (f \vee 0)$$
$$f^{-1} = (-f \vee 0)$$

Now $\int f \, dP = \int f^{+} \, dP - \int f^{-} \, dP$ whenever atleast one of the integral in RHS is finite.

Fatou's lemma

If $f_n \geq 0$ are measurable

Then $\liminf \int_{\Omega} f_n \, dP \geq \int_{\Omega} (\liminf f_n) \, dP$

$\liminf f_n = \sup_{k\geq 1} \left(\inf_{n \geq k} f_n \right)$ g_k

Note : $g_k \leq f_n \quad \forall\ n \geq k$

$\therefore \int g_k \, dP \leq \int f_n \, dP \quad \forall\ n \geq k$

Now, $\lim_{k \to \infty} \int_\Omega g_k \, dP = \int_\Omega \left(\lim_{k\to\infty} g_k \right) dP$

$= \int_\Omega (\liminf f_n)\, dP$

$\therefore \int g_k \, dP \leq \inf_{n \geq k} \int f_n \, dP$

$\Rightarrow \lim_{k\to\infty} \int g_k \, dP \leq \lim_{k\to\infty} \inf_{n\geq k} \int f_n \, dP$

$\Rightarrow \int_\Omega (\liminf f_n)\, dP \leq \liminf \int_\Omega f_n \, dP.$

Dominated Convergence Theorem

If $|f_n| \leq g$ & $f_n \to f$ all measurable & g is integrable (i.e $\int |g| \, dP < \infty$)

Then

(i) $\int f_n \, dP \longrightarrow \int f \, dP$

(ii) $\int |f_n - f| \, dP \longrightarrow 0.$

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<u>Expectations</u>

Let $(\Omega, \mathcal{F}, P)$ be a probability space

Define a random variable $X : \Omega \to \mathbb{R}$

Then $E[X] = \int_{\Omega} X \, dP$

& $E[X^2] = \int_{\Omega} X^2 \, dP$ wherever they exist

Consider $(P \circ X^{-1})((-\infty, x])$

$= P(X^{-1}(-\infty, x])$

$= P(\omega : X(\omega) \leq x)$

$= P(X \leq x) = F(x)$

This $F(x)$ is called the distribution function.

<u>Properties of distribution function</u>

(i) F is right continuous.

(ii) F is non-decreasing

(iii) $F(x) \to 1$ as $x \to \infty$

& $F(x) \to 0$ as $x \to -\infty$

$$\int_{\Omega} X(\omega) \, dP = \int_{\mathbb{R}} x \, d(P \circ X^{-1})((-\infty, x])$$

$$= \int x \, dF(x).$$

Bounded Variation

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have BV property on $[a, b]$ if

$$\max_{\substack{\{t_i\} \\ a = t_0 < t_1 < \dots < t_n = b}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty$$

Prof. Gopal Basak

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Probability Theory - Lecture 4

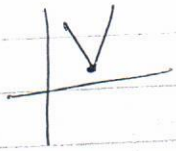
Absolutely continuous functions

$\forall \ \varepsilon > 0, \ \exists \ \delta > 0$ s.t $\sum_{j=1}^{n} (t_j - t_{j-1}) < \delta$

$\Rightarrow \sum_{j=1}^{n} | f(t_j) - f(t_{j-1}) | < \varepsilon$

where $t_0 < t_1 < \ldots < t_n$

Absolutely continuous functions $\Rightarrow$ Bounded variation.

Q. Check whether  is absolutely continuous or not.

Q. Check where $f(x) = \begin{cases} \sqrt{|x|} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

is absolutely continuous on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Conditional Expectation

Let $(\Omega, \mathcal{F}, P)$ be the probability space.

$\mathcal{G}$ is a σ-field s.t $\mathcal{G} \subseteq \mathcal{F}$

Consider $\min\limits_{f \text{ is } \mathcal{G}\text{-measurable}} E[X - f]^2 \ \cancel{\text{s.t. } E[f] = E[X]}$

Then $\arg\min f = E[X \mid \mathcal{G}]$

where

$$\int_A E(X \mid \mathcal{G})\, dP = \int_A X\, dP \quad \forall A \in \mathcal{G}$$

Proof: $E[X - E[X \mid \mathcal{G}]]^2 = E[X - f + f - E[X \mid \mathcal{G}]]^2$

$$= E[X - f)^2 + E[f - E(X \mid \mathcal{G})]^2 + 2E[(X - f)(f - E(X \mid \mathcal{G}))]$$

$$E[(X-f)(f - E(X \mid \mathcal{G}))] = E[E[(X-f)(f - E[X \mid \mathcal{G}])] \mid \mathcal{G}]$$

Since f $\mathcal{G}$-measurable & h $\mathcal{F}$ measurable $\Rightarrow$
$E[fh \mid \mathcal{G}] = f\, E[h \mid \mathcal{G}]$

$\therefore$ We get

$$E[(X-f)(f - E(X \mid \mathcal{G}))] = E[(f - E[X \mid \mathcal{G}])]\, E((X - f) \mid \mathcal{G})$$

$$= E[f - E(X \mid \mathcal{G})]^2$$

$$\therefore E[X - E[X \mid \mathcal{G}]]^2 = E[X - f]^2 - E[f - E[X \mid \mathcal{G}]]^2$$

$$\leq E[X - f]^2$$

let $\mathcal{G} = \{\phi, \Omega\}$

Then $E[X | \mathcal{G}] = E(X)$ a.s.
& only measurable functions of $\mathcal{G}$ are constants.

$$c\, P(\Omega) = \int_{\Omega} c \, dP = \cancel{\int_{\Omega} E(X|\mathcal{G})\, dP}$$
$$= \int_{\Omega} X \, dP$$
$$= E[X]$$
$$\therefore c = E[X].$$

let $X = 1_A$

$$\cancel{\int_B E[X|\mathcal{G}]\, dP} \cancel{=} \cancel{\int X \, dP}$$

let $\mathcal{G} = \{\phi, B, B^c, \Omega\}$

$$f = \begin{cases} c_1 & \text{on } B \\ c_2 & \text{on } B^c \end{cases}$$

let $X = 1_A$

$$\int_B E[X|\mathcal{G}]\, dP = \int_B X \, dP = \int_B 1_A \, dP = \int 1_A 1_B \, dP$$
$$= \int 1_{A\cap B} \, dP$$
$$= P(A\cap B).$$

$$\int_B (c_1 1_B + c_2 1_{B^c})\, dP = c_1 P(B\cap B) + c_2 P(B \cap B^c)$$
$$\Rightarrow c_1 P(B) = P(A\cap B)$$

$$\Rightarrow c_1 = P(A|B).$$

$$\therefore E(X|\mathcal{G}) = \begin{cases} P(A|B) & \text{for } \omega \in B \\ P(A|B^c) & \text{for } \omega \in B^c \end{cases}$$

In general, if $\mathcal{G} = \sigma(\{B_n\})$ where $\{B_n\}$ are partitions of Ω.

Then $f(\omega) = C_n$ on $\omega \in B_n$ are the only measurable functions.

Now, if X is a simple function,

$$X = \sum_i a_i 1_{A_i}$$

$$\Rightarrow E[X|\mathcal{G}) = \sum a_i E[1_{A_i}|\mathcal{G}]$$

$$= \sum a_i P(A_i | B_n) \text{ for } \omega \in B_n \text{ a.s.}$$

$$\Rightarrow E[X| B_n] = \frac{\sum a_i \int_{B_n} 1_{A_i} \, dP}{P(B_n)}$$

$$= \frac{\int_{B_n} X \, dP}{P(B_n)}.$$

For non-negative X, take over simple functions
For general X, write $X = X^+ - X^-$

Martingales

Let $\{X_n\}_{n \geq 0}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ satisfying

(i) $E(|X_n|) < \infty \quad \forall n$

(ii) $E(X_{n+1} \mid X_0, \ldots, X_n) = X_n$.

Then $\{X_n\}$ is called a martingale

Filtrations

$\mathcal{F}_1 = \sigma(X_1) \in \mathcal{B}$.

$\mathcal{F}_2 = \sigma(X_1, X_2) = \sigma(X_1^{-1}(B_1) \cap X_2^{-1}(B_2) : B_1, B_2 \in \mathcal{B}(\mathbb{R}$

Then choosing $B_2 = \Omega$ gives ~~elements~~ sets from $\sigma(X_1)$
& choosing $B_1 = \Omega$ gives from $\sigma(X_2)$

$\vdots$

$\therefore \quad \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \ldots$

Q. Show that if $S_n = X_0 + \ldots + X_n$, $P(X_i = 1) = p$ & $P(X_i = -1) = 1-p$
Then $\{S_n\}$ is a martingale w.r.t $\mathcal{F}_n = \sigma(X_0, \ldots, X_n$
when $p = q = \frac{1}{2}$.

Q. $W_n = S_n^2 - n$

Then $\{W_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

Q. $Z_n = \left(\frac{q}{p}\right)^{S_n}$ $p \neq q$

Then Z_n is a martingale w.r.t $\{\mathcal{F}_n\}$

Q. $Y_n = S_n - n(p-q)$ is a martingale w.r.t $\{\mathcal{F}_n\}$.

Q. $H_n = \exp\left\{S_n - \frac{n\sigma^2}{2}\right\}$ where $X_i \sim N(0, \sigma^2)$.

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Probability Theory – Lecture 5

$E(X|\mathcal{G})$ is the best (mean square deviation sense) estimator of X given the info of $\mathcal{G}$.

$\{X_n\}_{n\geq 0}$ is a martingale w.r.t $\{\mathcal{F}_n\}$, a filtration if it satisfies (i) $E|X_n| < \infty \quad \forall n$

(ii) $E(X_{n+1} | \mathcal{F}_n) = X_n$

Here, $\{\mathcal{F}_n\}$ is an increasing sequence of σ-fields & X_n is measurable w.r.t $\mathcal{F}_n$

$\therefore$ X_0 measurable w.r.t $\mathcal{F}_0$

X_1 " " $\mathcal{F}_1$

$\vdots$

X_k " " $\mathcal{F}_k$

But $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_k \subseteq \dots$

$\therefore$ X_0 measurable w.r.t $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_k$

eg. $\{S_n\}$ is a simple random walk

$P(X_i = +1) = p$,

$P(X_i = -1) = 1-p.$

$S_n = X_0 + \dots + X_n$

where $X_0 = c$

(i) $E|S_n| \leq E|X_0| + \dots + E|X_n|$

$= c + n < \infty$ for each n.

(ii) $E(S_{n+1} \mid X_0, \ldots, X_n) = E(S_n + X_{n+1} \mid \mathcal{F}_n)$.

$= E(S_n \mid \mathcal{F}_n) + E(X_{n+1} \mid \mathcal{F}_n)$.

$= S_n + 0$ ($\because$ in first part, S_n is measurable w.r.t $\mathcal{F}_n$. So it comes out of expectation & X_{n+1}. In second part X_{n+1} is independent of $\mathcal{F}_n$ $\therefore E(X_{n+1} \mid \mathcal{F}_n) = E(X_{n+1}) = 0$).

$= S_n$.

eg. $Z_n = \left(\frac{q}{p}\right)^{S_n}$

(i) $E|Z_n| = E\left(\frac{q}{p}\right)^{X_0 + \ldots + X_n}$

$= \left(\frac{q}{p}\right)^{c} E\left(\frac{q}{p}\right)^{X_1} \ldots E\left(\frac{q}{p}\right)^{X_n}$

Now $E\left(\frac{q}{p}\right)^{X_1} = \left(\frac{q}{p}\right) p + \left(\frac{q}{p}\right)^{-1} q$

$= q + p$
$= 1$

$\therefore E|Z_n| = \left(\frac{q}{p}\right)^{c}$

(ii) $E(Z_{n+1} \mid \mathcal{F}_n) = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} \mid \mathcal{F}_n\right]$.

$$= E\left[\left(\frac{q}{p}\right)^{S_n + X_{n+1}} \,\middle|\, \mathcal{F}_n\right]$$

$$= E\left[\left(\frac{q}{p}\right)^{S_n}\left(\frac{q}{p}\right)^{X_{n+1}} \,\middle|\, \mathcal{F}_n\right]$$

$$= \left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{X_{n+1}} \,\middle|\, \mathcal{F}_n\right]$$

$$= \left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right]$$

$$= \left(\frac{q}{p}\right)^{S_n}$$

$$= Z_n.$$

Gambler's Ruin

Start with Rs 20

$P(\text{win}) = p$

$P(\text{lose}) = 1-p$

$X_i = \pm 1$ in the i^{th} game

We want to know if we reach Rs 100 before reaching Rs. 0.

ie if τ is a stop time, then we want to find

$\phi(20) = P_{20}(\tau_{100} < \tau_0)$.

τ is a stopping time if $\tau : \Omega \to \{0, 1, \ldots, \infty\}$ satisfying $\{\tau \leq n\} \in \mathcal{F}_n$.

Generalising this,

$$\phi(c) = P_c(\tau_b < \tau_a) \qquad a < c < b.$$

$$\begin{tikzcd} & c+1 \\ c \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & c-1 \end{tikzcd}$$

$$\phi(c) = p\,\phi(c+1) + q\,\phi(c-1)$$

$$\Rightarrow p[\phi(c+1) - \phi(c)] = q[\phi(c) - \phi(c-1))$$

We solve this difference equation using boundary conditions

$$\phi(a) = 0$$
$$\phi(b) = 1.$$

<u>Doob Optional Stopping time</u>

Let $\{Y_n\}$ be a martingale w.r.t $\{\mathcal{F}_n\}$ & τ be a stopping time w.r.t $\{\mathcal{F}_n\}$.

Then $E[Y_\tau] = E[Y_0]$ or

$E[Y_\tau \mid \mathcal{F}_0] = Y_0$

provided

(i) $P(\tau < \infty) = 1$

(ii) $E|Y_\tau| < \infty$

(iii) $E(Y_m \mathbb{1}_{\tau > m}) \to 0$ as $m \to \infty$

Consider $E(Y_{n+1} \mid \mathcal{F}_n) = Y_n$

$$\Rightarrow E[E[Y_{n+1} \mid \mathcal{F}_n] \mid \mathcal{F}_{n-1}] = E[Y_n \mid \mathcal{F}_{n-1}]$$

$$\Rightarrow E[Y_{n+1} \mid \mathcal{F}_{n-1}] = Y_{n-1}$$

$\vdots$

$$E[Y_{n+1} \mid \mathcal{F}_0] = Y_0.$$

Q. Show that hypothesis of Doob's OST holds for Gambler's ruin.

If $p = q = \frac{1}{2}$, use $Y_n = S_n$

$$\tau = \tau_a \wedge \tau_b$$

Then $c = E_c(S_\tau) = a P_c(\tau_a < \tau_b) + b P_c(\tau_b < \tau_a)$

$$= a(1 - P_c(\tau_b < \tau_a)) + b P_c(\tau_b < \tau_a)$$

$$= (b-a) P_c(\tau_b < \tau_a) + a$$

$$\Rightarrow P_c(\tau_b < \tau_a) = \frac{c-a}{b-a}$$

If $p \neq q$, we use $Y_n = Z_n = \left(\frac{q}{p}\right)^{S_n}$

$$E_c(Z_\tau) = \left(\frac{q}{p}\right)^c$$

$$\Rightarrow \left(\frac{q}{p}\right)^a P_c(\tau_a < \tau_b) + \left(\frac{q}{p}\right)^b P_c(\tau_b < \tau_a) = \left(\frac{q}{p}\right)^c$$

$$\Rightarrow P_c(\tau_b < \tau_a) = \frac{\left(\frac{q}{p}\right)^c - \left(\frac{q}{p}\right)^a}{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^a}$$

If $E[Y_{n+1} \mid \mathcal{F}_n] \geq Y_n$
$\{Y_n\}$ is called submartingale

If $E[Y_{n+1} \mid \mathcal{F}_n] \leq Y_n$
$\{Y_n\}$ is called supermartingale.

Properties

1. If $\{Y_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ & ϕ is a convex function, then $\{\phi(Y_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ provided $\phi(Y_n)$ is properly defined & $E|\phi(Y_n)| < \infty$.

Similarly Concave functions give supermartingale.

2. Let $\{Y_n\}$ be a martingale w.r.t $\{\mathcal{F}_n\}$ & $\{A_n\}$ be a predictable sequence, i.e A_n is measurable w.r.t $\mathcal{F}_{n-1}$.

Define $X_n = Y_n - Y_{n-1}$ (called martingale difference sequence)

Then $Z_n = \sum_{k=1}^{n} A_k X_k$ is a martingale w.r.t $\{\mathcal{F}_n\}$

provided $E[A_k] < \infty$.

This can be considered as a discrete space integration where X_k denotes the difference.

Doob - Meyer Decomposition

Let $\{H_n\}$ be a submartingale w.r.t $\{\mathcal{F}_n\}$. Then $\exists$ $\{A_n\}$ an increasing predictable & integrable sequence s.t

$H_n - A_n = M_n$ is a martingale

i.e $H_n = M_n + A_n$ and the decomposition is unique with $A_0 = 0$.

$\left(A_n = \sum_{k=1}^{n} \left(E(H_k \mid \mathcal{F}_{k-1}) - H_{k-1}\right) \text{ works}\right)$.

Submartingale convergence theorem

Let $\{Y_n\}$ be a submartingale w.r.t $\{\mathcal{F}_n\}$ &

$\sup_n E(|Y_n|) < \infty$

Then $\exists\ Y_\infty$ s.t $Y_n \to Y_\infty$ as $n \to \infty$. a.s.

s.t $E(|Y_\infty|) < \infty$.

If $E[Y_n] \to E[Y_\infty]$, then $E|Y_n - Y_\infty| \longrightarrow 0$.

If $\{Y_n\}$ is a martingale itself then

$Y_n = E(Y_\infty | \mathcal{F}_n)$ a.s.

1/7/2015 2 p.m Prof. Gopal Ba

Probability Theory – Lecture 6

Markov Processes

$\{X_n\}_{n \in \mathbb{N}}$ is a discrete time process

$\{X_t\}_{t \in T}$ is a continuous time process.

A stochastic process $\{X_n\}_{n \geq 0}$ is called a Markov chain if

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0, \dots, X_n = x_n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

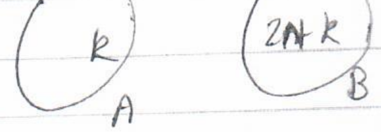
$\forall\ x_0, \dots, x_n, x_{n+1}$

whenever $P(X_0 = x_0, \dots, X_n = x_n) > 0$.

Transition is homogeneous if $P(X_{n+1} = y \mid X_n = x)$
$= P(X_1 = y \mid X_0 = x)$ $\forall n$

eg. Let S be the state space of a Markov Chain and it is finite.

$S = \{0, 1, \dots, 2N\}$



Let X_n = No. of molecules in A at time n.

~~If a mo~~ The experiment is to randomly pick a molecule. If it is in A, then transfer it to B. If it is in B, then transfer it to A

$$\therefore P(X_{n+1} = y \mid X_n = x) = \begin{cases} \dfrac{2N - x}{2N} & , y = x+1 \\ \dfrac{x}{2N} & , y = x-1 \\ 0 & , \text{otherwise.} \end{cases}$$

If π is the equilibrium distribution of this transition ma
& the k-step transition matrix is P^k

Then $\dfrac{1}{n+1} \sum_{k=0}^{n} P^k \longrightarrow \begin{bmatrix} \leftarrow \pi \rightarrow \\ \leftarrow \pi \rightarrow \\ \vdots \\ \leftarrow \pi \rightarrow \end{bmatrix}$

& $\pi P = \pi$

$\Rightarrow \pi_j = \sum_i \pi_i p_{ij}$

π is that distribution that makes X_i-distribution the same $\forall i$ if π is X_0-distribution.

$$\begin{aligned} P(X_1 = j) &= \sum_i P(X_1 = j, X_0 = i) \\ &= \sum_i P(X_0 = i)\, P(X_1 = j \mid X_0 = i) \\ &= \sum_i \pi_i p_{ij} \\ &= \pi_j \end{aligned}$$

Hence π is also called stationary / invariant distribu

Theorem: Let $\{X_n\}$ be an irreducible Markov Chain on a finite state space with period 1.

Then $p_{ij}^{(n)} \longrightarrow \pi_j$ as $n \to \infty$

Period: $d(i) = \gcd\{n : p_{ii}^{(n)} > 0\}$.

If $i \leftrightarrow j$, then $d(i) = d(j)$. Thus for an irreduci Markov Chain, there is only one period.

The reason why we took average in the previou example was that the Markov chain had period 2 $\therefore$ $p^{(n)}$ does not converge.

But the average will always converge.

For a simple random walk,

$$\sum_{n=0}^{\infty} p_{00}^{(n)} = \infty \text{ iff } p = q = \tfrac{1}{2} \longrightarrow \text{Called recurre}$$

$$< \infty \text{ iff } p \neq q \longrightarrow \text{Called transient.}$$

In an irreducible chain, all classes have similar properties.

Let $E_i(\tau_i)$ be the expected time to come t state i starting from i.

If $E_i(\tau_i) < \infty \rightarrow$ positive recurrent
$= \infty \rightarrow$ null recurrent.

In a random walk with $p = q = \frac{1}{2}$

$$\sum_{n=0}^{m} p_{00}^{(2n)} = \sum_{n=0}^{m} \binom{2n}{n} p^n q^n$$

$$= \sum_{n=0}^{m} \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \approx \sum_{n=0}^{n} \frac{1}{\sqrt{\pi n}} = \infty$$

By Sterling approx

In the continuous state sense, then $\{X_n\}$ is called irreducible if $\forall (x,y)$ in $S \times S$, $\exists\, n$ s.t $p^n(x,y) > 0$ where $p(x,y)$ is the density (note : <u>not</u> probability).

$$p^2(x,y) = \int_S p(x,z)\, p(z,y)\, dz.$$

In continuous time, Markov property is

$$P(X_t \in U \mid X_{r_k} \in U_{r_k}, \ldots), \quad r_1 < r_2 < \ldots$$
$$= P(X_t \in U \mid X_{r_k} \in U_{r_k})$$

whenever $P(X_{r_i} \in U_{r_i},\ i = 1, \ldots, k)$
$\forall\ r_1 < \ldots < r_k,\ k \geq 1$

$\therefore$ If X_s is measurable w.r.t $\mathcal{F}_s$, then $E(f(X_t) \mid \mathcal{F}_s)$ s.t $s < t$
$= E(f(X_t) \mid \sigma(X_s))$. if $\{X_s\}$ follows Markov property.

where $\mathcal{F}_s = \sigma(X_u : 0 \leq u \leq s)$.

Brownian Motion

$\{B_t\}_{t \geq 0}$ is a Brownian Motion if

(i) $t \rightarrow B_t$ is continuous.
$B_0 = x$ (or $B_0 = 0$) with P_x probability 1.

(ii) $(B_t - B_s) | \mathcal{F}_s$, $0 \leq s < t$ is independent of $\mathcal{F}_s$
i.e the increments are independent.

(iii) $B_t - B_s \sim N(\mu(t-s), \sigma^2(t-s))$.

For $\mu = 0$ & $\sigma = 1$, $\{B_t\}$ is called Standard Brownian motion.